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MONTHLY NOTICES

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E. J. Stone, M.A., F.R.S., President, in the Chair.

James Lawson, Houghton House, Ealing;
Emanuel Ristori, 13 Pemberton Crescent, Bayswater; and
Walter George Woollcombe, B.A., The College, Brighton;

were balloted for and duly elected Fellows of the Society.

An Explanation of the Principal Cause of the large Errors at present existing between the Positions of the Moon deduced from Hansen's Tables and from Observation; and the Cause of an Apparent Increase in the Secular Acceleration in the Moon's Mean Motion required by the Tables, or of an Apparent Change in the Time of the Earth's Rotation. By E. J. Stone, M.A., F.R.S.

The errors in the Lunar Theory have been traced to the effects of changes in the *unit of time*, or in the point of reference from which our theoretical longitudes are measured, which have, apparently unconsciously, been introduced from time to time into astronomy with changes in the adopted data.

The argument is clearly seen by a consideration of the different expressions for the longitudes of what may be called the mean Sun which have been adopted for the determination of the sidereal times at mean noon.

If B, H, and V denote the longitudes of the mean Sun, according to Bessel, Hansen, and Le Verrier, we have for 1850 Jan. 1, Paris mean noon $+t$.

$$B = 280^\circ\ 46'\ 36''{\cdot}12 + 1296027{\cdot}618184\,.\,t + 0{\cdot}0001221805\,.\,t^2$$

$$H = 280\ \ 46\ \ 43{\cdot}20 + 1296027{\cdot}674055\,.\,t + 0{\cdot}0001106850\,.\,t^2$$

$$V = 280\ \ 46\ \ 43{\cdot}51 + 1296027{\cdot}678400\,.\,t + 0{\cdot}0001107300\,.\,t^2$$

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In all these expressions the unit of time has been *supposed* to be a Julian year of 365·25 mean solar days.

The constant differences, 7″·08 and 7″·39, in B − H and B − V are not unimportant, for they introduce abrupt changes in the record of time; but the differences in the coefficients of t and t^2 show that the *same* unit of time cannot have been adopted in these expressions. The measure of time must be continuous, and the point from which our R.As. are measured, definite. Let, therefore, 1 and $(1+x)$ be the units of time in B and H. Then

$$1296027{\cdot}618184 \,.\, t + 0{\cdot}0001221805\, t^2$$
$$= 1296027{\cdot}674055 \,.\, t\,(1+x) + 0{\cdot}0001106850 \,.\, t^2(1+x)^2.$$

If, therefore,

$$n' = 1296027{\cdot}674055$$
$$x = -\frac{0{\cdot}055871}{n'} + \frac{0{\cdot}000114955 \,.\, t}{n'}.$$

To reconcile B and H, therefore, x must contain a variable term. Similar remarks apply to the difference between B and V.

Now let N be the Moon's mean motion referred to 1 as the unit of time, and $(N+\delta N)$ the Moon's mean motion referred to $(1+x)$ as the unit of time. Then

$$(N+\delta N)\,(1+x) = N,$$

and

$$t \,.\, \delta N = \frac{N}{n'}\{0{\cdot}055871 \,.\, t - 0{\cdot}0000114955 \,.\, t^2\} = 0''{\cdot}747 \,.\, t - 1''{\cdot}54\left(\frac{t}{100}\right)^2.$$

But Hansen determined his mean motion of the Moon so as to force an agreement between his theory and observations reduced with Bessel's unit 1, and his Tables therefore represented the observations well for many years whilst 1 was adopted as the unit of time; but directly the unit of time was changed by the adoption either of H or V, then the effects of the erroneous determination of the Moon's mean motion by Hansen become apparent.

The effect of the change of unit is also shown in the comparison of Le Verrier's Solar Tables with observation, but of course only to about the thirteenth part of the amount shown by the Lunar Tables.

Le Verrier's Tables have been employed in the computation of the Sun's positions in the *Nautical Almanac* since 1864; the Greenwich observations available to me only extend to 1880. The following table exhibits the mean excess in longitude of Le Verrier's tabular places over those deduced from observation. It will be seen that the errors are clearly progressive, and the change per annum appears roughly to be about +0″·052, a result confirming, I think, my opinion of the undesirability of the increase of the mean motion of the Sun by +0″·063.

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Year.	Mean Excess in Longitude.	No. of Obs.	Year.	Mean Excess in Longitude.	No. of Obs.
1864	+0″·025	122	1873	+0″·408	124
1865	+0·136	222	1874	+0·762	106
1866	−0·109	109	1875	+0·728	110
1867	−0·203	93	1876	+0·675	112
1868	+0·245	131	1877	+0·683	87
1869	+0·272	98	1878	+0·816	85
1870	+0·122	126	1879	+0·748	77
1871	+0·306	113	1880	+0·861	115
1872	−0·037	120			

Table showing the agreement of Hansen's Tables with actual observation before *the change in the unit of time, and the progressive error* after *the change in the unit of time introduced in* 1864.

Year.	Mean Value for Year (H−O).	No. of Obs.	Year.	Mean Value for Year (H−O).	No. of Obs.
1847	+1″·066	107	1855	−0″·871	94
1848	+0·195	113	1856	−0·956	93
1849	−0·465	109	1857	−1·858	115
1850	−0·282	114	1858	−1·975	130
1851	−1·295	109	1860	not compared	
1852	−0·917	109	1861	,,	
1853	−1·629	105	1862	−2·829	85
1854	−1·684	121	1863	1·606	115

Here change in the unit of time took place.

Year.	Mean Value for Year (H−O).	No. of Obs.	Year.	Mean Value for Year (H−O).	No. of Obs.
1864	+0·121	111	1873	+8·239	94
1865	+1·271	108	1874	+9·294	117
1866	+2·142	107	1875	+9·867	94
1867	+3·480	89	1876	+9·755	82
1868	+4·117	125	1877	+9·234	105
1869	+4·277	94	1878	+8·219	90
1870	+4·828	102	1879	+9·631	85
1871	+6·955	106	1880	+10·265	101
1872	+7·309	109			

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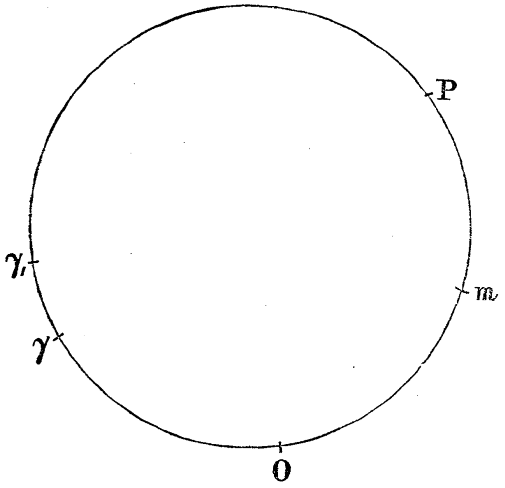
To get rid of the effects of the diurnal rotation I have referred everything to the meridian of some Observatory, as Paris, and discussed the changes of sidereal times at *mean* Paris noons; but as there is apparently some difficulty in following this reasoning, which is based upon the principle of continuity, I give a direct proof of the legitimacy of the method adopted in throwing the changes in the law fixing the position of the point from which our theoretical R.As. are measured, relatively to the Paris meridian at mean noon, on the time in the case of the mean Sun, from whose motion our ideas of mean solar time are derived, and on the determinations of the *mean* motions for the Moon and the planets.



If we assume the time of the Earth's rotation on its axis to be constant—an assumption to be tested by the results—we can always determine the position, relatively to our meridian, of a fixed point on the equator at any definite time. Let this point be O. This is a simplification merely to get rid of the complication of ideas arising from the diurnal motion.

Let m be the mean Sun by whose motion our time is to be measured, and at time t, expressed in any definite unit, let its position be at m, so that

$$\mathrm{O}m = nt$$

(n is a *definite* numerical quantity, since the rotation of the Earth on its axis is supposed to be assigned); and at the same time let P be a planet, and

$$\mathrm{OP} = \mathrm{A} + n't.$$

For practical convenience we measure our angles from γ, where

$$\mathrm{O}\gamma = \mathrm{C} + at + st^2,$$

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C, a, and s being assigned constants.

$$\therefore\ \gamma m = \mathrm{C} + (a + n)t + st^2$$
$$\gamma \mathrm{P} = \mathrm{C} + \mathrm{A} + (a + n')t + st^2.$$

But suppose, instead of referring our angles to γ, we refer them, in our theoretical investigations, to $\gamma_{,}$, such that

$$\mathrm{O}\gamma_{,} = \mathrm{C} + \delta\mathrm{C} + (a + \delta a)t + (s + \delta s)t^2;$$

then

$$(1)\quad \gamma_{,}m = \mathrm{C} + \delta\mathrm{C} + (a + n + \delta a)t + (s + \delta s)t^2,$$

t having the *same* value as before.

$$(2)\quad \gamma_{,}\mathrm{P} = \mathrm{C} + \delta\mathrm{C} + \mathrm{A} + (a + \delta a + n')t + (s + \delta s)t^2.$$

Now, if we know of the changes $\delta\mathrm{C}$, δa, δs, and allow for them, we shall correctly determine the relative motion of P measured from $\gamma_{,}$; but if we do not know of these changes, and do not allow for them, how shall we force an agreement between observation and theory? Clearly we shall have to throw the error on the time measured from (1), and on the mean motions of the planets in (2), and we shall thus partially force an agreement, but not a complete one, between observations referred to γ and theory referred to $\gamma_{,}$.

First, let t be the true time deduced either from γm or $\gamma_{,}m$; and let $(t+\delta t)$ be the observed time deduced from (1) when $\delta\mathrm{C}$, δa, and δs are neglected, or when $\gamma_{,}m$ is forced to agree with observations referred to γ; and let B be the epoch, x the mean motion, and yt^2 the empirical secular correction required to make $\gamma_{,}\mathrm{P}$, deduced with the time $(t+\delta t)$, agree with $\gamma\mathrm{P}$; we then have

$$\gamma_{,}\mathrm{P} = \mathrm{B} + x(t + \delta t) + (s + \delta s + y)t^2;$$
$$= \mathrm{C} + \mathrm{A} + (a + n')t + st^2.$$

But from (1) we have

$$\mathrm{O} = \delta\mathrm{C} + (a + n)\delta t + \delta a \,.\, t + \delta s \,.\, t^2$$
$$\delta t = \frac{-(\delta\mathrm{C} + \delta a \,.\, t + \delta s \,.\, t^2)}{a + n},$$

and

$$\gamma_{,}\mathrm{P} = \mathrm{B} - \frac{x\,.\,\delta\mathrm{C}}{a + n} + x\left(1 - \frac{\delta a}{a + n}\right)t + \left(s + y + \delta s - \frac{\delta s\,.\,x}{a + n}\right)t^2$$
$$= \mathrm{C} + \mathrm{A} + (a + n')\,.\,t + st^2$$
$$\therefore\ \mathrm{B} - \frac{x\,.\,\delta\mathrm{C}}{a + n} = \mathrm{C} + \mathrm{A}$$
$$x\left(1 - \frac{\delta a}{a + n}\right) = (a + n')$$

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$$s+y+\delta s-\frac{\delta s\,.\,x}{a+n}=s$$

$$\therefore\ x=(a+n')\left(1+\frac{\delta a}{a+n}\right)$$

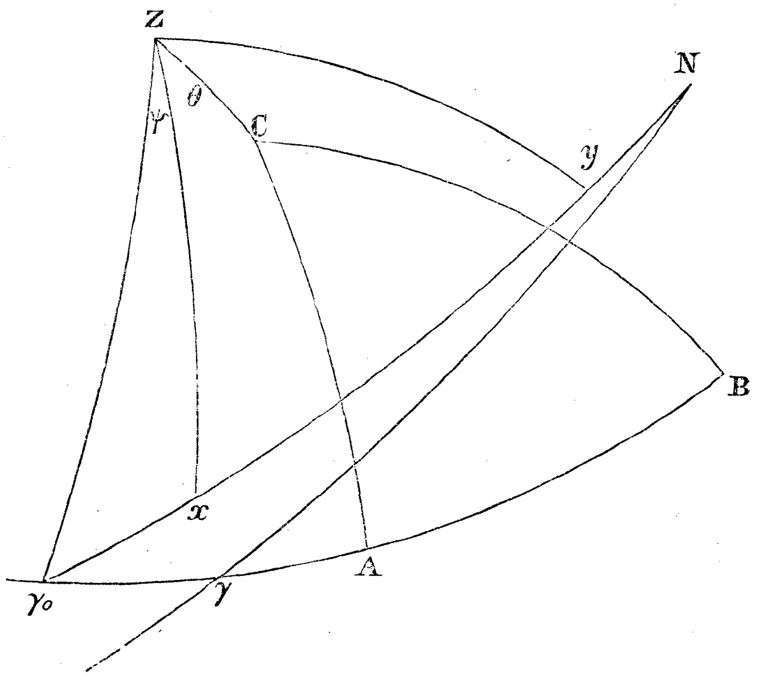
$$\mathrm{B}=\mathrm{C}+\mathrm{A}+\delta\,\mathrm{C}\left(\frac{a+n'}{a+n}\right)$$

$$y+\delta s=\delta s\left(\frac{a+n}{a+n}\right)$$

The correction required to γ P deduced directly from the erroneous time to make it agree with the observations referred to γ is therefore

$$-(\delta\,\mathrm{C}+\delta a\,.\,t+\delta s\,.\,t^2)\left(\frac{a+n'}{a+n}\right),$$

which is the condition upon which the investigation in my paper is based as deduced directly from the consideration of the differences in the units of time necessary to secure continuity in the adopted expressions for sidereal time at mean noon. The possibility of such errors arises from the indirect way in which our theories are compared with observation. Numerical values of the constants which fix the relative changes in the planes and points of reference and the mean motion of Sun have to be adopted and carried into the theories; but the observed R.As. are deduced from the times of transit by clocks whose errors are determined by the differences between the times of transit by the clock and R.As. measured from the point of intersection of the ecliptic and equator, which point must, for accurate work, be the same as that adopted in the theoretical expressions. I have shown, however, by the discordant values of B, H and V, that no definite point, much less the true point, has been practically adopted in our theories.



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The breaks of continuity in our measures which have necessarily taken place with the adoption of different units through the introduction of small changes in the Sun's mean notion, and in the secular terms which express the relative motions of the planes of reference, are clearly shown as follows:—

Let x, y, Z be the projections of three rectangular axes fixed in space, xy being the plane of the ecliptic and x the point of intersection of the ecliptic and equator at the instant from which our time is measured; and at the time t let γ_0AB be the plane of the equator, γN the plane of the ecliptic, C the projection of the polar axis of the Earth, and CA the meridian of the Paris Observatory.

Let

$$\mathrm{ZC}=\theta; \qquad \gamma_0 \mathrm{Z}x=\psi; \qquad \gamma_0 \mathrm{A}=\phi;$$

n = angular velocity of the Earth on its axis.

Then

$$n=\frac{d\phi}{dt}-\frac{d\psi}{dt}\,.\cos\theta.$$

This equation is exact.

And

$$\phi=\mathrm{C}+nt+\int\frac{d\psi}{dt}\,.\cos\theta\,.\,dt,$$

where C is some constant to be determined from observation. From the theory of the motion of the Earth about its centre of gravity, we have, to the second power of t,

$$(1)\ \ \psi=at+bt^2-\Psi.$$

$$(2)\ \ \theta=\omega_0+ft^2-\Omega:$$

where a, b, ω_0, f are constants, Ψ and Ω periodic functions.

$$\therefore\ \int\frac{d\psi}{dt}\,.\cos\theta\,.\,dt=at\cos\omega_0+bt^2\cos\omega_0+\Psi\,.\cos\omega_0,$$

and

$$\phi=\mathrm{C}+(n+a\cos\omega_0)\,.\,t+bt^2\cos\omega_0-\Psi\,.\cos\omega_0.$$

But from the relative motion of the Sun around the Earth, we have

$$(3)\ \ x\mathrm{N}=\alpha-\beta t; \qquad (4)\ \ \gamma_0\mathrm{N}\gamma=\gamma t+\epsilon t^2,$$

where α, β, γ, and ϵ are constants.

From (1), (2), (3), and (4) we obtain

$$\gamma_0\gamma=ut+u't^2;\ \text{where } u \text{ and } u' \text{ are constants.}$$

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But

$$\gamma A = \phi - \gamma_C \gamma = s, \text{ suppose};$$

$$s = C + (n + a \cos \omega_0 - u)t + (b \cos \omega_0 - u')t^2 + \Psi \cos \omega_0.$$

This equation fixes the law of variation of γ the point from which the R.As. are measured, relatively to A, a fixed point on the Earth's surface; and nothing but confusion can result from any alteration in this expression unless the changes in the constants by which it is produced are consistently carried through all our work.

Now suppose the mean longitude of the Sun

$$= L_0 + Nt + \lambda t^2.$$

Let

$$n + a \cos \omega_0 - u - N = 365{\cdot}25 \times 2\pi;$$

this fixes the unit of time where

n = angular velocity of Earth on its axis,
a = luni-solar precession,
ω_0 = obliquity at 1850, Jan. 1,
u = intercept on Equator between ecliptic-planes,
n' = Sun's sidereal mean motion in longitude,
p = general precession.
$N = n' + p$.

All these quantities, when expressed numerically, have to be referred to a Julian year of 365·25 mean solar days. But since the rotation of the Earth on its axis, and the Sun's mean motion in longitude are physical facts, we must have

$$\frac{n}{n'} = C,$$

where C is a constant to be determined from observation; and this relationship must hold good *whatever unit of time be adopted*, and be carried into the equation of condition which fixes the unit of time from the relative motions of the Earth on its axis, and the Sun's mean motion in R.A.

Similarly we may write

$$a \cos \omega_0 - u - p = An';$$

where A is some constant.

Our equation of condition which defines the unit of time then becomes

$$n'(C - 1 + A) = 365{\cdot}25 \times 2\pi. \qquad (2)$$

This equation shows at once that we cannot change n' without changing our unit of time. We can of course write

$$(n' + \delta n')\,(C - 1 + A) = 365{\cdot}25 \times 2\pi, \qquad (3)$$

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and still *call* the new unit of time a *Julian year*, but in this case the length of a Julian year in (3) is not the same as in (2); and if t_1 and t_2 are the lengths of the two so-called Julian years, we must have

$$t_1 n' = t_2(n' + \delta n'); \qquad (4)$$

therefore

$$t_1 = t_2\left(1 - \frac{\delta n'}{n'}\right). \qquad (5)$$

From which it directly follows that if a period be expressed by t Julian years under system (2), it will exceed the period expressed by t Julian years under system (3) by

$$\frac{\delta n'}{n'} \cdot t. \qquad (6)$$

I now proceed to show how it has been frequently assumed that the error due to a change $\delta n'$ is only

$$= \frac{\delta n'}{n} \cdot t, \text{ instead of } \frac{\delta n'}{n'} \cdot t.$$

Let n and $n + \delta n$ be the angular velocities of the Earth on its axis, corresponding to the two values n' and $(n' + \delta n')$ of the Sun's mean motion. Then

$$n = 365{\cdot}25 \times 2\pi + n' + p + u - a \cos \omega_0; \qquad (7)$$

but it is then usually assumed that

$$n + \delta n = 365{\cdot}25 \times 2\pi + n' + \delta n' + p + u - a \cos \omega_0, \qquad (8)$$

from which of course would follow

$$\delta n = \delta n',$$

and

$$\frac{\delta n}{n} = \frac{\delta n'}{n};$$

but upon this I must remark that since t_1 and t_2 cannot *both* be put $=1$, we must have instead of (8), the equation of condition

$$n + \delta n = 365{\cdot}25 \times 2\pi(1 + x) + n' + \delta n' + (1 + x)(p + u - a \cos \omega_0). \quad (9)$$

(The change of unit in n' is allowed for in $\delta n'$.)

$$\therefore\ \delta n = 365{\cdot}25 \times 2\pi x + \delta n' + x(p + u - a \cos \omega_0);$$

$$\therefore\ \delta n = \frac{\delta n'}{n'}(365{\cdot}25 \times 2\pi + n' + p + u - a \cos \omega_0);$$

$$= \frac{\delta n'}{n'} \cdot n;$$

$$\therefore\ \frac{\delta n}{n} = \frac{\delta n'}{n'};$$

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or the change of unit of time is as

$$1 : 1 + \frac{\delta n'}{n'},$$

as before proved.

Since the epoch from which t is measured is not yet fixed, let it be so chosen that

$$C = L_0.$$

Then

$$s = L_0 + Nt + 365{\cdot}25 \times 2\pi \times t + (b \cos \omega_0 - u')t^2 + \Psi . \cos \omega_0.$$

If the epoch thus chosen be called 1850, Jan. 1, Paris mean noon, we have for the computation of the distance γ from A at the different Paris mean noons,

$$S_0 = L_0 + Nt + (b . \cos \omega_0 - u') . t^2 + \Psi . \cos \omega_0.$$

The non-periodic part of S_0 only differs from the mean longitude of the Sun by

$$\{\lambda - (b . \cos \omega_0 - u')\} . t^2.$$

If, therefore, we call

$$L = L_0 + Nt + (b . \cos \omega_0 - u') . t^2$$

the longitude of the mean Sun, the mean longitude of the Sun

$$= L + \{\lambda - (b . \cos \omega_0 - u')\} . t^2.$$

Here

$$\{\lambda - (b \cos \omega_0 - u')\} . t^2$$

is a small secular term which, with the data adopted in Le Verrier's Tables—

$$= -0{\cdot}00003113 . t^2 = -0{\cdot}3113\left(\frac{t}{100}\right)^2.$$

If it should be found absolutely necessary, in the progress of science, to introduce corrections δN and $\delta\lambda$ to N and λ in order to represent the solar observations, we must keep

$$L = L_0 + Nt + (b \cos \omega_0 - u')t^2$$

still unchanged, but put the mean longitude of the Sun

$$= L + \delta Nt + (\delta\lambda - 0{\cdot}00003113) . t^2.$$

If the constants determining the motions of the planes of reference and the mean motion of the Sun are accurately known, we shall, in this way, have for a very long period the point of *reference* γ coincident with the point of intersection of the ecliptic and equator; but even if this condition is not quite satisfied, we must not change γ by changing the law of variation of Aγ until we deliberately change our unit of time, and carry

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logically through all our work the consequences of that change. If the difference $0{\cdot}3113\left(\frac{t}{100}\right)^2$ between the longitude of the mean Sun and the mean longitude of the Sun be neglected (and this has usually been done), it would lead to the apparent necessity of an increase of about 4″·16 in the secular acceleration of the Moon's mean motion. This quantity agrees very closely with the supposed discordance between theory and observation. It would appear, therefore, that the discordance between theory and observation which has recently been thrown on the Earth's rotation no longer exists to any proved sensible extent, and, so far as we know at present, the time of the Earth's rotation is constant.

[This abstract was drawn up for the Society's publications by request, after a verbal explanation of the paper, which was given at the meeting of April 13, 1883.]

On the Computation of the Eccentric Anomaly, Equation of the Centre and Radius Vector of a Planet, in Terms of the Mean Anomaly and Eccentricity. By J. Morrison, M.D., M.A., Assistant on the American *Ephemeris and Nautical Almanac*, Washington, D.C.

(*Communicated by the Foreign Secretary.*)

Several methods of computing the eccentric anomaly of a planet from the eccentricity and mean anomaly have already been given, in the *Monthly Notices* and elsewhere, by many very able mathematicians. Some of these methods involve considerable labour even after the formulæ have been deduced, while others give only rough approximations. The following method of treating the subject is due, I believe, to the late Professor Hansen, as indicated by him in the *Abhandlungen der Sächsischen Gesellschaft der Wissenschaften*, Band II. It gives the eccentric anomaly with great accuracy and facility by means of rapidly converging series, and in most cases with very little labour, especially after the coefficients of the several terms of the series have been computed for each planet. I purpose in this short paper to develop the subject more fully than is done in the work referred to, and to apply the results to the computation of the eccentric anomaly in the case of each of the primary planets, as well as of the equation of the centre and the radius vector.

Let

$\mathrm{M}=$ the Mean Anomaly,

$\mathrm{E}=$ the Eccentric Anomaly,

$\nu=$ the True Anomaly,

$e=$ the Eccentricity,

and

$\epsilon=$ the Napierian base·